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MINIMAX IMPULSIVE CORRECTION OF PERTURBATIONS OF A LINEAR DAMPED OSCILLATOR

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A minimax formulation is used to solve the problem of synthesizing an impulsive correction of the zero position of a linear damped oscillator subjected to a continuous perturbation. Analytic expressions are obtained for the corrective impulses and a system of transcendental equations defining their instants of application. The dependence of the processes considered on the dynamic properties of the oscillator is analysed, using the numerical data obtained.

1. Statement of the problem. We consider on a given time interval $[t_0, T]$ an impulse-controlled oscillator with nonnegative friction, subjected to a continuous perturbation

$$x^{\cdot\cdot} + 2\alpha x^{\cdot} + k^2 x = \sum_{k=1}^{n} u_k \delta(t - t_k) + v(t) \tag{1.1}$$

$$x(t_0) = x_0, \quad x^{\cdot}(t_0) = y_0$$

The corrective control is restricted by the number of impulses n and by their combined magnitude (correction resource)

$$\sum_{k=1}^{n} |u_k| \leqslant Q \tag{1.2}$$

The perturbation $v(t)$ can be represented by any arbitrary measurable function bounded in modulo by a known continuous positive function $p(t)$

$$|v(t)| \leqslant p(t), \quad t_0 \leqslant t \leqslant T \tag{1.3}$$

We pose the problem of synthesizing a control $\{u_k, t_k\}$, restricted by (1.2) and $t_0 \leqslant t_1 \leqslant \ldots \leqslant t_n \leqslant T$, guaranteeing a minimum value of the final deviation of the position of the oscillator from its zero position, i.e. of the functional

$$J = |x(T)| \tag{1.4}$$

We assume that the phase state $(x, x^{\cdot})$ of the oscillator is known at every instant of time. The instants of application of the impulses $t_1, \ldots, t_n$ are chosen before the beginning of the process. The choice is based on the computation of the worst perturbation, and it must be reviewed during the process in accordance with the actual perturbation.

The problem stated above is a differential-impulse game [1]. The case $\alpha = k = 0$ was studied in [2].

During the analysis of the problem it is expedient to perform the change of variables

$$z(t) = e^{A(T-t)} \; y(t)$$

where $y = (x, x^{\cdot})$ is the vector of the phase state of the system (1.1), and e^{At} is the fundamental matrix of solutions of the corresponding normal homogeneous system. From the form of the functional (1.4) it follows that it is sufficient to consider only the first coordinate of the vector z which coincides at the end of the process with the value $x(T)$. Below we shall denote this coordinate simply by the letter z without any indices.

Thus, in the case of real eigenvalues $\lambda_{1,2} = -\alpha \mp \sqrt{\alpha^2 - k^2}$ we consider the variable

$$z(t) = \frac{\lambda_2 e^{\lambda_1(T-t)} - \lambda_1 e^{\lambda_2(T-t)}}{\lambda_2 - \lambda_1} x(t) - \frac{e^{\lambda_1(T-t)} - e^{\lambda_2(T-t)}}{\lambda_2 - \lambda_1} x^{\cdot}(t), \quad |k| < \alpha \qquad (1.5)$$

$$z(t) = [(1 + \alpha(T - t))\, x(t) + (T - t)\, x^{\cdot}(t)]\, e^{-\alpha(T-t)}, \qquad |k| = \alpha \quad (1.6)$$

and in the case of complex values $\lambda_{1,2} = -\alpha \mp i\beta$ where $\beta = \sqrt{k^2 - \alpha^2}$, the variable

$$z(t) = [(\alpha \sin\beta(T - t) + \beta\cos\beta(T - t))\, x(t) + x^{\cdot}(t) \sin\beta(T - t)]\, e^{-\alpha(T-t)} / \beta \qquad (1.7)$$

By virtue of (1.1) and (1.2), the variable $z(t)$ is defined by the equations

$$z^{\cdot} = \varphi(t)\, v, \quad z(t_k^+) = z(t_k^-) + \varphi(t_k)\, u_k, \qquad k = 1, \ldots, n \qquad (1.8)$$

$$\varphi(t) = \begin{cases} \dfrac{e^{\lambda_2(T-t)} - e^{\lambda_1(T-t)}}{\lambda_2 - \lambda_1}, & |k| < \alpha \\ (T - t)\, e^{-\alpha(T-t)}, & |k| = \alpha \\ e^{-\alpha(T-t)} \sin\beta(T - t)/\beta, & |k| > \alpha \end{cases}$$

The initial condition for this variable can be found from one of the relations (1.5) — (1.7) by setting $t = t_0$, $x = x_0$, $x^{\cdot} = y_0$.

2. Solution of the problem for arbitrary values of $t_1, \ldots, t_n$. The problem of synthesizing the optimal correction (strategy of the controlling player) for arbitrary values of $t_1, \ldots, t_n$ is solved using the method of dynamic programing. The guaranteed value of the functional $J = |z(T)|$ for $(n - k + 1)$-step process which begins at the instant t_k before the action of the impulse u_k, is determined by the state $z(t_k^-) = z_k^-$, i.e. the residual correction resource

$$q_k = \begin{cases} Q, & k = 1 \\ Q - \sum_{i=1}^{k-1} |u_i|, & k = 2, \ldots, n \end{cases}$$

and by the values $t_k, \ldots, t_n$. We shall denote this quantity by $S_{n-k+1}(z_k^-, q_k, t_k, \ldots, t_n)$. The value of the stated game is the function

$$\omega_n(z_0, Q, t_1, \ldots, t_n) = \max_{v[t_0, t_1]} S_n(z_1^-, Q, t_1, \ldots, t_n) \qquad (2.1)$$

The recurrent equations (2.2) and the terminal condition (2.3) follow from the principle of optimality

$$S_{n-k+1}(z_k^-, q_k, t_k, \ldots, t_n) = \qquad (2.2)$$

$$\min_{u_k} \max_{v[t_k, t_{k+1}]} S_{n-k}(z_{k+1}^-, q_{k+1}, t_{k+1}, \ldots, t_n), \quad k = 1, \ldots, n-1$$

$$S_1(z_n^-, q_n, t_n) = \min_{u_n} \max_{v[t_n, T]} |z(T)| \qquad (2.3)$$

Computations of the extrema in (2.1)—(2.3) are stipulated by the restrictions $|u_k| \leqslant q_k$, (1.3) and the relations (1.8). From (2.3) we obtain the equality

$$S_1(z_n^-, q_n, t_n) = \max\{0, |z_n^-| - q_n\varphi(t_n)\} + \int_{t_n}^{T} p(t)|\varphi(t)|\,dt$$

which shows that the function S_1 does not decrease with increasing $|z_n^-|$, and the solution of (2.2) for $k = n-1$ reduces to finding the quantity

$$\min_{u_{n-1}} \max_{v[t_{n-1}, t_n]} |z_n^-|$$

The solution of this problem is equivalent to finding the extrema in (2.3) and shows, that the function S_2 does not decrease with increasing $|z_n^-|$ either, etc. Thus all equations of (2.2) and (2.1) have the same solution, the worst interference has the form

$$v(t) = p(t)\,\mathrm{sign}\, z(t_j^+), \quad t_j \leqslant t < t_{j+1}, \; j = 0, 1, \ldots, n \qquad (2.4)$$

and the optimal correction impulses are described by the equations

$$u_k = -\mathrm{sign}\, z_k^- \min\{q_k, \; |z_k^-| / |\varphi(t_k)|\}, \quad k = 1, \ldots, n \qquad (2.5)$$

The impulse u_k is called the compensating impulse when the system under consideration arrives, in the absence of perturbations on the interval (t_k, T), to the state $x(T) = 0$ at the instant T. This is clearly equivalent to the equation $z(t_k^+) = 0$.

The relations (2.5) imply directly that all impulses, except perhaps the last one, are compensating ones. Thus, if the optimal correction consists of m nonzero impulses $(m \leqslant n)$, then

$$|u_k| = |z_k^-| / \varphi(t_k)|, \qquad k = 1, \ldots, m-1, \quad |u_m| \leqslant q_m \qquad (2.6)$$

and in an extremal process, i.e. in a process with the interference (2.4) and impulses (2.5), the following relations are satisfied:

$$|z_1^-| = |z_0| + F(t_0, t_1), \quad |z_j^-| = F(t_{j-1}, t_j), \quad j = 2, \ldots, m \qquad (2.7)$$

$$F(t', t'') = \int_{t'}^{t''} p(t)|\varphi(t)|\,dt$$

Moreover, the guaranteed value of the functional (1.4) is expressed by the equation

$$J = \begin{cases} F(t_m, T), & q_m \geqslant |z_m^-| / |\varphi(t_m)| \\ |z_{m-1}^+| + F(t_{m-1}, T) - q_m|\varphi(t_m)|, & q_m < \dfrac{|z_m^-|}{|\varphi(t_m)|} \end{cases} \qquad (2.8)$$

From the compensating property of the impulses $u_1, \ldots, u_{m-1}$ it follows that $z_{m-1}^+ = 0$ when $m \geqslant 2$. The expression (2.8) obviously represents also the functions S_k, $k = 1, \ldots, m$ corresponding to the extremal process, and the value of the game is $\omega_m(z_0, Q, t_1, \ldots, t_m)$.

3. Optimization of the instants of application of the impulses. Solution of (2.1) — (2.3) gives a synthesis of the worst interference (2.4), and of the optimal correction (2.6) corresponding to the arbitrary times t_k of the application of the impulses u_k. To find the optimal values $t_1, \ldots, t_m$, we must minimize the function $\omega_m(z_0, Q, t_1, \ldots, t_m)$ with respect to the variables sought.

We see from (1.8) and (2.6) that the effectiveness of the correction time t_k is determined by the value of the function $|\varphi(t)|$ at this point. In the aperiodic case ($|k| \leqslant \alpha$) the function $\varphi(t)$ is positive and decreases monotonously on the interval $[t_0, T]$. In the oscillatory case ($|k| > \alpha$) this function possesses the above properties on the interval $[\tau, T)$, where the point

$$\tau = T - \frac{1}{\beta}\operatorname{arctg}\frac{\beta}{\alpha}$$

represents the absolute maximum of the function $|\varphi(t)|$ on the interval $(-\infty, T]$. In both cases we have $\varphi(T) = 0$.

The above properties of the function $\varphi(t)$ lead to the assertion that the optimal correction must be carried out on the interval $[\theta, T)$, where

$$\theta = \begin{cases} t_0, & |k| \leqslant \alpha \\ \max\{t_0, \tau\}, & |k| > \alpha \end{cases}$$

The above assertion becomes trivial when $\theta = t_0$. Let $\theta > t_0$ (in the oscillatory case) and $t_0 \leqslant t_l < \tau \leqslant t_{l+1}$ $(1 \leqslant l \leqslant m)$. From (2.4) and (2.6) it follows that $|z(\tau^-)| = |z_{l-1}^+| + F(t_{l-1}, \tau) - |u_l \varphi(t_l)|$. If the impulse $|u_l|$ is transferred at the instant $t_l' = \tau$, then for $\tau < t_{l+1}$, by virtue of the inequality $|\varphi(t_l)| < \varphi(\tau)$ the quantity $|z(\tau^+)|$ is reduced in value without additional use of the correction resource, and in the case of $\tau = t_{l+1}$ the same value of $|z(\tau^+)|$ is guaranteed without impairing the economical use of the resource. The same effect can be achieved by transferring all impulses preceding the instant τ to the interval $[\tau, T)$, and this proves the assertion.

The optimal value of t_m must minimize the function (2.8) on the interval $[t_{m-1}, T]$. It is easily seen that this function is unimodal with respect to t_m and, in the case of $m > 1$, has a minimum at the point defined by the equation

$$q_m \varphi(t) = F(t_{m-1}, t), \quad t_{m-1} \leqslant t \leqslant T$$

Here the last correction is a compensating one, uses up the remaining resource q_m, and the quantity (2.8) assumes the value

$$J = F(t_m, T) \tag{3.1}$$

Execution of two or more optimal impulses in the worst case (2.4) is possible if the resource Q exceeds in magnitude the quantity necessary for carrying out a compensating correction at the most effective instant of time θ, i.e. when

$$Q > [|z_0| + F(t_0, \theta)] / \varphi(\theta) \tag{3.2}$$

Otherwise we must carry out a single correction at the instant θ using the whole of the resource Q. This guarantees a final deviation which does not exceed

$$J = |z_0| + F(t_0, T) - Q\varphi(\theta)$$

Below we consider a case in which the condition (3.2) is satisfied.

If we admit a unique corrective impulse ($n = 1$), the optimal value t_1 minimizing the quantity (2.8) on the interval $[\theta, T]$ is a solution of the equation

$$Q\varphi(t) = |z_0| + F(t_0, t)$$

Such a correction is also a compensating one and the guaranteed value of the functional is described by (3.1) in which $m = 1$.

Let the resource q_m of the last correction be distributed over two compensating impulses u_m' and u_{m+1} so that $|u_m'| = \mu q_m$ and $|u_{m+1}| = (1-\mu) q_m$, where $0 \leqslant \mu \leqslant 1$. The time instances t_m' and t_{m+1} are determined as functions of the parameter μ by the equations

$$F(t_{m-1}, t_m') = \mu q_m \varphi(t_m'), \quad F(t_m', t_{m+1}) = (1-\mu) q_m \varphi(t_{m+1})$$

From the above equations it follows that $t_m'(0) = t_{m-1}$, $t_m'(1) = t_{m+1}(0) = t_{m+1}(1) = t_m$ and the derivatives of the functions $t_m'(\mu)$ and $t_{m+1}(\mu)$ have the form

$$dt_m'(\mu) / d\mu = \Phi(t_m', \mu)$$

$$\frac{dt_{m+1}(\mu)}{d\mu} = \left[\frac{p(t_m')\varphi(t_m')}{q_m\varphi(t_{m+1})}\Phi(t_m', \mu) - 1\right]\Phi(_{m+1}, 1-\mu)$$

$$\Phi(t, \mu) = q_m\varphi(t) / [p(t)\varphi(t) - \mu q_m d\varphi / dt]$$

We clearly have

$$\left.\frac{dt_{m+1}}{d\mu}\right|_{\mu=0} = \left[\frac{\varphi(t_{m-1})}{\varphi(t_m)} - 1\right]\Phi(t_m, 1) \tag{3.3}$$

The function $\varphi(t)$ is strictly decreasing on the interval $[t_{m+1}, T]$ and the value of $\Phi(t_m, 1)$ is equal to the value of the derivative of $t_m'(\mu)$ at the point $\mu = 1$. This derivative is obviously positive for all $\mu \in [0, 1]$, therefore the expression (3.3) is positive and the function $t_{m+1}(\mu)$ has a maximum on the interval (0, 1) which exceeds the value of t_m.

The above analysis shows that the number of impulses should be increased to maximum, i.e. for the optimal correction we have $m = n$.

Thus, the optimal correction consists of n compensating impulses using up the whole resource Q and distributed on the interval $[\theta, T)$. When the worst interference (2.4) acts, the above statement is expressed, on the strength of Eqs.(2.6) which hold for $k = 1, \ldots, n$, and of the relations (2.7), by the following condition (3.4) and the inequalities (3.5):

$$|z_0| / \varphi(t_1) + \sum_{k=1}^{n} F(t_{k-1}, t_k) / \varphi(t_k) = Q \tag{3.4}$$

$$\theta \leqslant t_1 < \ldots < t_n < T \tag{3.5}$$

Thus, the problem of choosing the optimal instants of application of the corrective impulses is reduced to that of minimizing the function $F(t_n, T)$ under the conditions (3.4), with $t_0 \leqslant t_1 \leqslant \ldots \leqslant t_n \leqslant T$. The solution of this problem has the properties (3.5).

4. Realization of the correction. In order to generalize the procedure of solving the problem of correction of real objects and of accounting for the perturbations taking place, it is expedient to reduce the description of the object to the dimensionless quantities and zero initial condition. If at the instant t the object is in the state $z(t)$, then we can assume that the motion had begun at the instant of time t^* from its initial zero state, and was executed under the action of the worst interference. This value of t^* is obviously given by the equation $F(t^*, t) = |z(t)|$. The function $p(t)$ is additionally defined for $t < t_0$ by the quantity $p(t_0)$. The passage to the dimensionless quantities t', z', α', k', Q', v' and u_k' is carried out according to the formulas

$$t = (T - t^*)t' + t^*, \quad z = F(t^*, T)z' \tag{4.1}$$

$$v = p^* v', \quad u_k = p^* u_k', \quad Q = p^* Q'$$
$$\alpha = \alpha' / (T - t^*), \quad k = k' / (T - T^*), \quad p^* = F(t^*, T) / \int_{t^*}^{T} |\varphi(t)| \, dt$$

It can be confirmed that the above transformation reduces the initial problem to that with a unit time interval, zero initial conditions and, in the case of $p(t) = \text{const}$, with a unit perturbation intensity. The functions $\varphi(t)$, $p(t)$ and $F(t, T)$ become $\varphi'(t')$, $p'(t')$ and $F'(t', 1)$ with all the properties used above preserved, and the results previously obtained remain valid for the new problem. The optimal set of the times of application of the impulses is an internal point of the restricting set $0 \leqslant t_1' \leqslant \ldots \leqslant t_n' \leqslant 1$, therefore the problem of minimizing the function $F'(t', 1)$ on this set under the condition

$$\sum_{k=1}^{n} F'(t_{k-1}', t_k') / \varphi'(t_k') = Q' \tag{4.2}$$

can be solved using the Lagrange method. The necessary conditions for an extremum obtained in this manner for the unknown variables, consist of Eq. (4.2) and the system

$$\varphi'(t_{k+1}') = A(t_{k-1}', t_k') \varphi'(t_k'), \quad k = 1, \ldots, n-1 \tag{4.3}$$

where

$$A(t_{k-1}', t_k') = \frac{p'(t_k') [\varphi'(t_k')]^2}{p'(t') [\varphi'(t_k')]^2 - F(t_{k-1}', t_k') \, d\varphi'(t_k') / dt'}$$

It was shown in Sect. 3 (inequalities (3.5)) that the optimal values of the times of application of the impulses are distributed on the interval $[\theta', 1)$. The function $\varphi'(t')$ is positive on this interval and decreases monotonously, therefore $0 < A(t_{k-1}, t_k') \leqslant 1$ for $\theta' \leqslant t_k' < 1$ and $A(t_{k-1}, t_k') = 1$ only when $t_k' = \theta'$. From this it follows that the system (4.3) has a unique solution $t_k'(s)$, $k = 2, \ldots, n$ for every value of $t_1' = s \in [\theta', 1)$ with $t_k'(\theta') = \theta'$, $k = 1, \ldots, n$. Moreover, if $\theta' < s < 1$, then $s = t_1' < t_2' < \ldots < t_n' < 1$.

Let us now consider the left-hand part of Eq. (4.2) as a function of $q(s)$ obtained by setting $t_k' = t_k'(s)$. The values of this function are easily obtained by solving consecutively Eqs. (4.3). To find the parameter s generating the optimal set $\{t_k'(s)\}$, we must solve the equation

$$q(s) = Q', \quad \theta' < s < 1 \tag{4.4}$$

If this equation has more than one solution, then the optimal set of the corresponding collection of sets $t_1'(s), \ldots, t_n'(s)$ is the set with the largest value of $t_n'(s)$.

When a fictitious beginning of the process has to be introduced (i.e. when we have a nonzero initial condition), it may happen that the procedure described above yields a value of t_1 which is smaller than the initial t_0. In this case we have two possibilities, if all roots of Eq. (4.4) coinciding with the supposedly optimal values of t_1' lie to the left left of t_0', then the first correction ought to be carried out at the beginning of the process, i.e. $t_1 = t_0$, otherwise the best root of (4.4) chosen from amongst those lying on the interval $[t_0', 1)$ must be equated with the value $t_1' = t_0'$. To find the values $t_2', \ldots, t_n'$ when $t_1' = t_0'$ (or $t_1 = t_0$), it is sufficient to pass to the problem with zero initial condition, the number of impulses equal to $n - 1$ and the resource $Q_1 = Q - |z_0| / |\varphi(t_0)|$.

We used a numerical method of finding the optimal instants of time $t_1', \ldots, t_n'$ described above, for the case of $p(t) = \text{const}$ and for various values of α', k', Q' and n. The functional relations $t_2'(s), \ldots, t_n'(s)$ were found to be monotonously increasing, consequently the solutions of the system (4.2),(4.3) are unique.

The relations $t_1'(Q')$ and $J(Q')$ obtained for each parameter pair (α', k') are analogous to the corresponding relations given in [2] for the case $\alpha' = k' = 0$. The only essential difference is present in the oscillatory case for $\theta > 0$. Here, as it was shown in Sect. 3, a unique corrective impulse at the instant θ is applied for the values of the correction resource from the interval $[0, F(0, \theta) / \varphi(\theta)]$ and the relationship $J(Q')$ is linear.

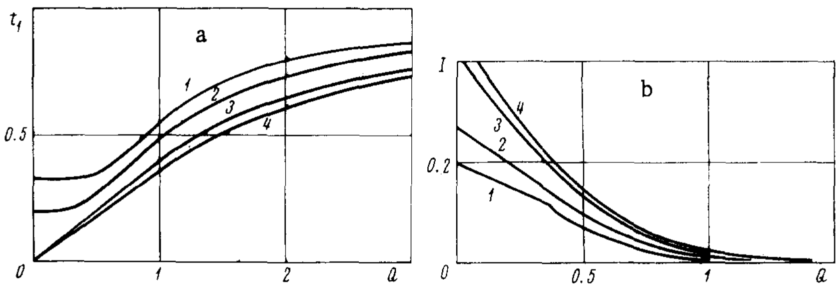

Fig. 1

The curves shown in Fig. 1 correspond to a three-impulse correction and to the following parameters: for the curves *1*, *2* and *3*, $k' = 1.5$ and $\alpha' = 1.49$, 0.75 and 0, for curve *4*, $k' = 1$ and $\alpha' = 0$, respectively. As we see, the guaranteed final deviation increases with decreasing friction α'.

Other numerical results show that when the quantity $\max\{\alpha', k'\}$ decreases, the curves $J(Q')$ in all cases move upwards and approach the corresponding curves given in [2]. If $\max\{\alpha', k'\}$ increases, the oscillator becomes less susceptible to perturbations (with respect to the functional) and the curves $J(Q')$ approach the abscissa axis.

We can improve the guaranteed value of the functional of the problem when the perturbation deviates from its worst mode, using the following idealized algorithm for synthesizing the instants of time of application of the corrective impulses. Let us assume that m impulses remain to be applied and the perturbation deviates from its worst mode on a certain interval before previously computed instant of application of the first of the remaining impulses. This produces a deviation of the quantity $|z(t)|$ from the value corresponding to the worst mode, and the quantities $t_1, \ldots, t_m$ are recomputed using the procedure described above. This recomputation is carried out for every instant of the interval in question and ends either when the running time coincides with the improved value of t_1, or when the worst interference is restored. After applying the consecutive impulse, the process is repeated with a correspondingly diminished correction resource and the number of impulses.

For the practical synthesis of the instants of time of application of impulses, an algorithm is possible, which differs from the proposed one in the frequency of the times of recomputing the values of $t_1, \ldots, t_m$. The frequency interval must not be less than the time necessary for description of the computation.

An approximate synthesis of the instants of time $t_1, \ldots, t_n$ can be realized with the help of the previously computed relationships $t_n'(Q')$ corresponding to the impulse numbers $n, n-1, \ldots, 1$ and a sufficiently dense network of parameters α_i' and k_i'. The latter parameters must be connected by the relation $\alpha k_i' = \alpha_i' k$, since the transformation (4. 1) changes them proportionally. Moreover when the perturbation deviates from its worst mode, the fictitious initial time t^* increases and this leads, by virtue of (4. 1), to decrease in the values of α' and k'. Thus the network of parameter α' (or k') represents a decomposition of the segment $[0, (T - t_0^*)\alpha]$, where t_0^* is a fictitious initial time corresponding to the initial conditions of the problem.

In this manner, the initial correction problem which has, in the case of a constant intensity, eight parameters x_0, y_0, T, α, k, p, Q and n is reduced, by means of transformation (4. 1), to a problem with four parameters α', k', Q' and n. Thus a synthesis of a correction for a real object requires only n one-parameter (α' or k') relations $t_1'(Q')$.

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QUASI-NORMAL AND NORMAL OSCILLATIONS IN CONSERVATIVE SYSTEMS

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Particular kinds of periodic solutions in unison — quasi-normal oscillations similar to those of an oscillator — are separated in conservative multidimensional systems. A new definition of normal oscillations, more precise than known ones is proposed. It is applicable to a wider class of nonlinear systems. A method of approximate determination of quasi-normal oscillations for a particular kind of nonlinear systems is described and some examples are presented.

In [1, 2] the supposition was made that singular analogs of characteristic solutions, often called normal oscillations, can exist in the class of nonlinear conservative systems of the form $x''_i = \partial U / \partial x_i$, $U(0) = 0$, $U(-x) = U(x)$ and $x = \{x_1, x_2, \ldots, x_n\}$. It was assumed that normal oscillations are determined by the following characteristic properties: oscillation frequencies of all coordinates are equal, all coordinates attain their maximum deflection and vanish simultaneously, and the displacement of coordinates at any instant of time is a single-valued function of one of these.

From the physical point of view the above definition of normal oscillations has the following shortcomings: the characteristic properties of normal oscillations are noninvariant under the change of the coordinate system, are interdependent, comprise a narrow class of nonlinear systems, and do not permit the formulation of the problem of determining normal oscillations.

In this paper the concept of normal oscillations of nonlinear systems is extended,